Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-10-18

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

(a) The literal translation of the inequality is $r^2 + z^2 \leq R^2$. If we want to sweep out the region only once, we must further impose the inequalities $r \geq 0$ and $0 \leq \theta \leq 2\pi$. If you were to set up an actual integral, you could write

$$\int_0^{2\pi} \int_0^R \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \ldots r \, dz \, dr \, d\theta.$$

where the function $(\ldots)$ has been omitted.

(b) The literal translation of the inequality is $\rho^2 \leq R^2$. If we want to sweep out the region only once, we must further impose the inequalities $\rho \geq 0$, $0 \leq 2\pi \leq \theta$, and $0 \leq \phi \leq \pi$. If you were to set up an actual integral, you could write

$$\int_0^{2\pi} \int_0^{\pi} \int_0^R \ldots \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Question 2. We talked about how to systematically do this in class, so here are just the answers.

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} 1 \, dz \, dy \, dx$$

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} 1 \, dz \, dx \, dy$$

$$\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} 1 \, dy \, dz \, dx$$

$$\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} 1 \, dy \, dx \, dz$$

$$\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \, dz \, dy$$

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \, dy \, dz$$

Question 3. This is a cone of height H whose base is a circle of radius R. Its vertex is at the origin, and it is upside-down.

Answers to computations

Problem 1. This is a straightforward integral so I'll omit the computations, but the answer is $\frac{1}{3}\pi R^2 H$.

Problem 2. We just have to put the function 1 in place of $(\ldots)$ in the integrals from Question 1. In spherical, all the integration is quite easy. In cylindrical, you will want to use a change of variables such as $u = R^2 - r^2$, $du = -2r \, dr$ for the middle integral.

Problem 3. The sphere is unfortunately not the graph of any function, but we can break it up into two hemispheres. The surface area of the entire sphere is double the surface area of

$$z = \sqrt{R^2 - x^2 - y^2}.$$

Setting up the integral as per §15.5, the hemisphere has surface area

$$\iint_D \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}} \, dx \, dy$$

where D is the disk $x^2 + y^2 \leq R^2$. This is a good indicator that we ought to switch to polar:

$$\int_0^{2\pi} \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} \, dr \, d\theta.$$

For the inner integral, perform the change of variables $u = R^2 - r^2$, $\mathrm{d}u = -2r\,\mathrm{d}r$. The rest is straightforward integration. Don't forget to double your answer at the end!

As for the observation that $\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2$, this is because as we increase R by an infinitesimal amount dR, the volume is increased by a thin spherical shell of thickness $\mathrm{d}R$. The volume of this infinitesimally thin spherical shell is $\mathrm{d}R$ times the surface area of the sphere.